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COMPACTIFICATIONS OF LOCALLY COMPACT GROUPS AND CLOSED SUBGROUPS

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ABSTRACT. Let G be a locally compact group with closed normal subgroup N such that G/N is compact. In this paper, we construct various semigroup compactifications of G from compactifications of N of the same type. This enables us to obtain specific information about the structure of the compactification of G from the structure of the compactification of N. Our results seem to be interesting and new even when G is the additive group of real numbers and N is the integers. Applications and other examples are given.

1. INTRODUCTION

A key technique in the study of locally compact groups has been to "induce" properties of a group from information about its closed subgroups. This has been dramatically successful in the theory of group representations (Mackey [22], Dixmier [10], Kirillov [20]). In a less exalted sphere, there are theories allowing the extension of functions in various classes from a subgroup to the whole group (de Leeuw and Glicksberg [9], Dixmier [10], Henrichs [17], Berglund et al. [4]). In this paper, we shall also obtain results of the latter kind; though they will not be new, they will appear as corollaries to a new theory. Our principal concern is with semigroup compactifications ($\mathscr{LC}$, $\mathscr{WAP}$, $\mathscr{AP}$, etc.) of a group G and we show how to construct these from the same compactifications of a closed normal subgroup N provided that G/N is compact and that some further conditions are satisfied (which always are when G is commutative). As corollaries, besides results about extension of functions, we also find information about the structure of compactifications of G.

We shall indicate the scope of our theory by describing its application in a special case, when $G = \mathbb{R}$ and $N = \mathbb{Z}$. Let $I = [0, 1]$; then $I + \mathbb{Z} = \mathbb{R}$. (More details are given in 5.3.) We interpret this as saying that $\mathbb{R}$ can be obtained from $\mathbb{Z}$ by attaching a copy of I between each pair of points n, $n+1$ ($n \in \mathbb{Z}$). If we take a universal compactification of $\mathbb{Z}$, say $\mathbb{Z}^{\mathscr{WAP}}$ (the weakly almost periodic compactification of $\mathbb{Z}$), then our theory says that $\mathbb{R}^{\mathscr{WAP}}$ can be obtained by adjoining I between each pair x, $x+1$ ($x \in \mathbb{Z}^{\mathscr{WAP}}$). The same holds for other compactifications. Minimal left ideals in, say, $\mathbb{R}^{\mathscr{LC}}$ can be obtained

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from minimal left ideals in $\mathbb{Z}^{\mathscr{LC}}$ ($= \beta\mathbb{Z}$) by the same process. Results similar to these for $\mathscr{LC}$-compactifications have also been obtained recently by Filali [13].

To discuss functions, given $x \in \mathbb{R}$, we write $x = t + n$ with $t \in I$, $n \in \mathbb{Z}$. We shall find that $f \in \mathscr{WAP}(\mathbb{R})$ if and only if both $n \to f(t+n)$ is in $\mathscr{WAP}(\mathbb{Z})$ for each $t \in I$, and the set of functions $\{t \to f(t+n) \mid n \in \mathbb{Z}\}$ is equicontinuous in $\mathscr{C}(I)$. The analogous result holds for $\mathscr{LC}$, $\mathscr{AP}$, and so on.

This paper is organised as follows. In §2 we give some basic definitions and notations. In §3 we construct a locally compact semigroup from a semigroup compactification X of a closed normal subgroup N of a locally compact group G; it is obtained as a quotient $(G \times X)/\rho$ of a semidirect product $G \times X$. When G/N is compact, $(G \times X)/\rho$ is shown to be a semigroup compactification of G. In §4 the results of §3 are applied to characterize functions of various kinds on G in terms of functions of the same kind on N. §5 is devoted to examples.

2. Preliminaries

For notation and terminology we shall follow Berglund et al. [4, especially, or 3] as much as possible. Thus a *topological semigroup* is a semigroup S that is also a Hausdorff topological space, multiplication $(s, t) \to st: S \times S \to S$ being continuous. S is a *semitopological semigroup* if multiplication is only separately continuous, i.e., the maps $s \to st$ and $s \to ts$ from S into S are continuous for each $s \in S$. For S to be *right topological*, only $s \to ts$ is required to be continuous. The spaces of almost periodic, weakly almost periodic, left continuous, distal, point distal and minimal ($\mathbb{C}$-valued) functions are denoted by $\mathscr{AP}$, $\mathscr{WAP}$, $\mathscr{LC}$, $\mathscr{D}$, $\mathscr{PD}$ and $\mathscr{MIN}$, respectively. The reader is reminded that, for a topological group G, $\mathscr{LC}$ is just the space of bounded functions uniformly continuous with respect to the right uniformity of G [18, p. 21]. Also, the first four of these spaces are left m-introverted; $\mathscr{PD}$ and $\mathscr{MIN}$ are not left m-introverted, in general, but $\mathscr{PD}$ is a translation invariant C^*-subalgebra of $\mathscr{LC}$, while $\mathscr{MIN}$, although translation invariant, need not be a linear space. Other references for these spaces are: de Leeuw and Glicksberg [8] and Burckel [5] for $\mathscr{AP}$ and $\mathscr{WAP}$; Mitchell [26] for $\mathscr{LC}$ (which is called $\mathscr{LUC}$ in [26] and elsewhere); and J. Auslander and Hahn[1], L. Auslander and Hahn [2], Flor [14] and Knapp [21] for $\mathscr{D}$, $\mathscr{PD}$ and $\mathscr{MIN}$.

Throughout this paper G will denote a locally compact group with identity e. A *semigroup compactification* of G is a pair (ψ, X), where X is a compact right topological semigroup with identity 1 and $\psi: G \to X$ is a continuous homomorphism with $\psi(G)^- = X$ and

$$\psi(G) \subset \Lambda(X) := \{x \in X \mid y \to yx,\ X \to X \text{ is continuous}\}.$$

$\Lambda(X)$ is called the *topological center* of X. When no confusion can arise, we often refer to (ψ, X), or even to X, as a *compactification* of G. The reader is directed to [4, §3.1] for the correspondence between compactifications and left m-introverted subalgebras of $\mathscr{C}(G)$, and for a discussion of properties P of compactifications and associated universal mapping properties. (Other references for the concept of introversion are Day [7] and Mitchell [25].) For the purposes of illustration, we mention here only that $G^{\mathscr{WAP}}$, the weakly almost periodic compactification of G or the $\mathscr{WAP}$-compactification of G, is regarded as the spectrum of $\mathscr{WAP}(G)$. One property of compactifications (ψ, X) with respect to which $G^{\mathscr{WAP}}$ is universal is: X is a semitopological semigroup. Another is: $\psi^*(\mathscr{C}(X)) \subset \mathscr{WAP}(G)$. (See [8, §5, or 4 or 3].) In general, the compactification corresponding to a left m-introverted subalgebra $\mathscr{F}$ is $(\varepsilon, G^{\mathscr{F}})$, where $\varepsilon\colon G \to G^{\mathscr{F}}$ is the evaluation mapping; then $\varepsilon^*(\mathscr{C}(G^{\mathscr{F}})) = \mathscr{F}$.

A compactification (ψ, X) has the *joint continuity property* if the function

$$(s, x) \to \psi(s)x\colon (G \times X) \to X$$

is continuous. We state first a consequence of Ellis's Theorem [11, or 4, Theorem B.1].

2.1 **Lemma.** *Every compactification of a locally compact group G has the joint continuity property.*

Immediately from the definitions we get

2.2 **Lemma.** *If (ψ, X) is a compactification of G, then both $x \to x\psi(s)$ and $x \to \psi(s)x$ are continuous for $s \in G$.*

The next lemma collects some well-known results for easy reference.

2.3 **Surjectivity Lemma.** *Let S and T be semigroups and let $\theta\colon S \to T$ be a surjective homomorphism.*

(i) *Let each of S and T have a minimal left ideal and a minimal right ideal.*

(a) *If L is a minimal left ideal in S, $\theta(L)$ is a minimal left ideal in T.*

(b) *If R is a minimal right ideal in S, $\theta(R)$ is a minimal right ideal in T.*

(c) *If M is the minimal ideal in S, $\theta(M)$ is the minimal ideal of T.*

(d) *If G is a maximal group in the minimal ideal of S, $\theta(G)$ is a maximal group in the minimal ideal of T.*

Furthermore, the mappings $L \to \theta(L)$, etc., in (a)–(d) *of ideals in S to ideals in T or of groups in S to groups in T are surjective.*

(ii) *Let S and T have topologies with T compact, let θ be continuous, and suppose there is a compact $K \subseteq S$ with $\theta(K) = T$.*

(a) *If S is right topological, T is right topological.*

(b) *If S is left topological* (*analogously defined*), *T is left topological.*

(c) *If S is semitopological, T is semitopological.*

(d) *If S is topological, T is topological.*

(e) *If $s \to sx$ is continuous in S, $t \to t\theta(x)$ is continuous in T.*

(f) *If $s \to xs$ is continuous in S, $t \to \theta(x)t$ is continuous in T.*

Proof. (i)(a) Surjectivity implies that $\theta(L)$ is a left ideal. Let $L' \subseteq \theta(L)$ be a left ideal. Then $\theta^{-1}(L') \cap L$ is a left ideal contained in L, so is equal to L; thus $L \subseteq \theta^{-1}(L')$. So $\theta(L) \subseteq L'$, and $\theta(L) = L'$. To establish the last claim of (i), let L_1 be a minimal left ideal of T. Then $\theta^{-1}(L_1)$ is a left ideal of S, hence contains a minimal left ideal of S, which θ must map onto L_1.

The proof of (b) is similar. (c) follows as $M = LR$ for any minimal left ideal L and minimal right ideal R. (d) follows as $G = RL$ for some minimal left ideal L and minimal right ideal R.

(ii) We prove (e) first. Let $\{t_\alpha\} \subset T$ be a net converging to t. We prove any subnet of $\{t_\alpha\}$ has a further subnet which, when multiplied on the right by $\theta(x)$, converges to $t\theta(x)$ (and that finishes the proof). For each α take s_α in

K with $\theta(s_\alpha) = t_\alpha$. Given a subnet $\{s_\beta\}$ of $\{s_\alpha\}$, we can find a further subnet $\{s_\gamma\}$ converging to some $s \in K$. Then

$$t_\gamma\theta(x) = \theta(s_\gamma)\theta(x) = \theta(s_\gamma x) \to \theta(sx) = \theta(s)\theta(x) = t\theta(x).$$

The proof of (f) is similar. (a), (b) and (c) follow from (e) and (f). The proof of (d) is left to the reader. □

3. The quotient semigroup

Let G be a locally compact group with closed normal subgroup N, and let (ψ, X) be a compactification of N. As in Hahn [16, §5], we let ρ be the equivalence relation on $G\times X$ with equivalence classes $\{(sr^{-1}, \psi(r)x) \mid r \in N\}$. Thus

$$(s, x)\ \rho\ (t, y) \text{ if and only if } t^{-1}s \in N \text{ and } \psi(t^{-1}s)x = y.$$

π will denote the quotient map from $G\times X$ onto the quotient space $(G\times X)/\rho$. Clearly π is one-to-one on $\{e\}\times X$, so it is meaningful to identify $X \cong \{e\}\times X$ with its image in $(G\times X)/\rho$. We will often do this. It is important for this work that $(G\times X)/\rho$ is locally compact and Hausdorff. For completeness, we give proofs of these facts. The technical details are collected in the next lemma.

3.1 **Lemma.** (i) *The graph of ρ is closed.*

(ii) $\pi : G\times X \to (G\times X)/\rho$ *is an open mapping.*

(iii) $(G\times X)/\rho$ *is Hausdorff.*

(iv) *Let K be a compact subset of G and let $L = KN$. Then $\pi(K\times X)$ is compact and equals $\pi(L\times X)$.*

Proof. (i) To verify that the graph of ρ is closed, let $(s_\alpha, x_\alpha) \to (s, x)$ and $(t_\alpha, y_\alpha) \to (t, y)$ in $G\times X$ with $(s_\alpha, x_\alpha)\ \rho\ (t_\alpha, y_\alpha)$, i.e.,

$$t_\alpha^{-1}s_\alpha \in N \quad \text{and} \quad \psi(t_\alpha^{-1}s_\alpha)x_\alpha = y_\alpha,$$

for all α. Then $t_\alpha^{-1}s_\alpha \to t^{-1}s$, which is in N, since N is closed. The joint continuity property of X implies that $\psi(t_\alpha^{-1}s_\alpha)x_\alpha \to \psi(t^{-1}s)x$, hence $\psi(t^{-1}s)x = y$, as required.

(ii) To prove that π is open, let $O \subset G\times X$ be open. We must show that $A := \pi^{-1}(\pi(O))$, the union of the ρ-classes of the members of O, is open in $G\times X$. Let $(t, y) \in A$, $(t, y) = (sr^{-1}, \psi(r)x)$ for an $(s, x) \in O$ and $r \in N$. Let $V \subset G$ and $W \subset X$ be open and satisfy $(s, x) \in V\times W \subset O$. Then $Vr^{-1}\times \psi(r)W$ is open in $G\times X$, contains (t, y), and is contained in A.

(iii) Suppose that $P_1 = \pi(s, x)$ and $P_2 = \pi(t, y)$ are points of $(G\times X)/\rho$ such that every neighbourhood of P_1 meets every neighbourhood of P_2. We must show that $P_1 = P_2$. Since the openness of π implies, for example, that $\{\pi(U) \mid U \text{ is a neighbourhood of } (s, x)\}$ is a neighbourhood base for P_1, there exist nets $\{(s_\alpha, x_\alpha)\}$ and $\{(t_\alpha, y_\alpha)\}$ converging in $G\times X$ to (s, x) and (t, y), respectively, and also satisfying $(s_\alpha, x_\alpha)\ \rho\ (t_\alpha, y_\alpha)$ for all α. But the graph of ρ is closed, so $(s, x)\ \rho\ (t, y)$, i.e., $P_1 = P_2$, as required.

(iv) Since π is continuous and K is compact, $\pi(K\times X)$ is compact; it remains to show that $\pi(K\times X) = \pi(L\times X)$. But, if $(t, x) \in L\times X$, $t = sr$ for an $s \in K$ and $r \in N$, and $\pi(t, x) = \pi(s, \psi(r)x) \in \pi(K\times X)$. □

3.2 **Proposition.** *The quotient space* $(G\times X)/\rho$ *is locally compact and Hausdorff. If* G/N *is compact, then* $(G\times X)/\rho$ *is compact.*

Proof. Lemma 3.1(iii) asserts that $(G\times X)/\rho$ is Hausdorff. The local compactness of $(G\times X)/\rho$ is established by noting that $\pi(V\times X)$ is a compact neighbourhood of $\pi(s, x)$, if V is a compact neighbourhood of s.

The second claim follows from Lemma 3.1(iv) and the fact that compactness of G/N is equivalent to the existence of a compact $K\subset G$ such that $G=KN$; see [9, 5.24(b)]. □

Let $\mu: G\to G\times X$ be defined by $\mu(s)=(s, 1)$ (where 1 is the identity of X).

3.3 **Lemma.** $\pi\circ\mu$ *is a continuous map from* G *onto* $\pi(G\times\psi(N))$. *Furthermore, if* ψ *is a homeomorphism of* N *into* X, *then* $\pi\circ\mu$ *is also a homeomorphism; moreover,* G *is homeomorphic to an open subset of* $(G\times X)/\rho$.

Proof. As the composition of continuous maps, $\pi\circ\mu$ is continuous. It is onto; for, if $(s, \psi(r))\in G\times\psi(N)$, then $(s, \psi(r))\ \rho\ (sr, 1)$. Suppose ψ is injective. Then, if $(s, 1)\ \rho\ (t, 1)$, we have $\psi(t^{-1}se)=1$, i.e., $s=t$; so $\pi\circ\mu$ is injective. Suppose ψ is a homeomorphism of N into X. Then ψ is open, since N is locally compact. To prove that $\pi\circ\mu$ is also open, take open $V\subset G$. We must show that $W:=\pi^{-1}(\pi\circ\mu(V))$ is open in $G\times X$. But

$$\begin{aligned} W &= \{(s, x)\mid (s, x)\ \rho\ (t, 1) \text{ for some } t\in V\}\\ &= \{(s, x)\mid t^{-1}s\in N \text{ and } \psi(t^{-1}s)x=1 \text{ for some } t\in V\}. \end{aligned}$$

The equality in the last set implies that $x=\psi(s^{-1}t)$, and so W is just

$$\{(s, \psi(r))\in G\times\psi(N)\mid sr\in V\},$$

which is open in $G\times X$, since $\psi: N\to X$ is open. □

If N is a closed normal subgroup of G, we define $\sigma_s(r)=s^{-1}rs$ for $s\in G$ and $r\in N$. A compactification X of N is said to be *compatible* with G if each σ_s extends to a continuous function from X into X, i.e., if $\{\psi(\sigma_s(r_\alpha))\}$ converges in X whenever $\{\psi(r_\alpha)\}$ does. Compatibility implies that each σ_s determines a continuous transformation of X, for which we use the same notation σ_s. In 5.5, we present some examples of noncompatible compactifications.

3.4 **Lemma.** *Suppose that the compactification* X *of* N *is compatible with* G. *Then, for each* $s\in G$, σ_s *is a continuous automorphism of* X.

Proof. Since $\sigma_s(N)=N$, σ_s must be a homeomorphism of X onto X (with inverse $\sigma_{s^{-1}}$). We show σ_s is a homomorphism. First, we have

$$(*)\qquad \sigma_s(xy)=\sigma_s(x)\sigma_s(y)$$

for $x, y\in\psi(N)$. Since X is a right topological semigroup with $\psi(N)\subset\Lambda(X)$, we conclude that $(*)$ holds for $x\in\psi(N)$, $y\in X$, and then that $(*)$ holds for all $x, y\in X$, as required. □

If N is a closed normal subgroup of G and the compactification X of N is compatible with G, a semidirect product multiplication on $G\times X$ may be defined by

$$(s, x)(t, y)=(st, \sigma_t(x)y).$$

(This semidirect product formulation seems forced on us at this juncture, no doubt by the one-sided continuity of the setting. It is not the usual one, as in [4 and 18]. For other purposes in this paper we will follow [4 and 18], taking $G = G_1 \times G_2$, where G_2 acts on G_1 with the product given by

$$(s, t)(s_1, t_1) = (s\sigma_t(s_1), tt_1).)$$

3.5 **Lemma.** *Let G, N and X be as just described. Then $G \times X$ is a right topological semigroup. Also the map*

$$((s, r), (t, y)) \to (st, \psi(\sigma_t(r))y): (G \times N) \times (G \times X) \to G \times X$$

is continuous. Furthermore, the equivalence relation ρ is a congruence.

Proof. The continuity conclusions are easily established using Ellis's Theorem. We prove that ρ is a congruence. Suppose that $(s, x)\ \rho\ (t, y)$ and $(u, z) \in G \times X$. Then $t^{-1}s \in N$ and $\psi(t^{-1}s)x = y$, so that, for example,

$$(s, x)(u, z) = (su, \sigma_u(x)z)\ \rho\ (tu, \sigma_u(y)z) = (t, y)(u, z),$$

which follows because $(tu)^{-1}su = u^{-1}t^{-1}su \in N$ and

$$\psi((tu)^{-1}su)\sigma_u(x)z = \sigma_u(\psi(t^{-1}s)x)z = \sigma_u(y)z. \quad \square$$

3.6 **Theorem.** *Let N be a closed normal subgroup of G and let X be a compactification of N that is compatible with G. Then $(G \times X)/\rho$ is a locally compact, right topological semigroup, and a compactification of G if G/N is compact. Also, $(G \times X)/\rho$ has the following universal property. Let (φ, Y) be a semigroup compactification of G such that the restriction of φ to $N \subset G$ extends to a continuous homomorphism $\tilde{\varphi} : X \to Y$ in such a way that for each $s \in G$ and $x \in X$*

$$\tilde{\varphi}(\sigma_s(x)) = \varphi(s^{-1})\tilde{\varphi}(x)\varphi(s).$$

Then there is a (unique) continuous homomorphism $\vartheta: (G \times X)/\rho \to Y$ such that $\vartheta \circ \pi \circ \mu = \varphi$, where $\mu(s) = (s, 1) \in G \times X$ and $\pi : G \times X \to (G \times X)/\rho$ is the quotient map.

Proof. The first claims have been established in Lemmas 3.2–3.5.

To prove the universal property, we first define $\vartheta_0 : G \times X \to Y$ by $\vartheta_0(s, x) = \varphi(s)\tilde{\varphi}(x)$. ϑ_0 is continuous, since Y is a compactification of G, and also a homomorphism, as is readily verified. ϑ is obtained as a quotient of ϑ_0 by showing that ϑ_0 is constant on ρ-classes of $G \times X$. The verification of this is also left to the reader. $\square$

Let P be a property of compactifications which admits universal P-compactifications (at least for locally compact groups G). Let $\mathscr{P} = \mathscr{P}(G)$ be the left m-introverted subalgebra of $\mathscr{LC}(G)$ whose spectrum $G^{\mathscr{P}}$ is the universal P-compactification. (See [4], especially §3.3, for this.) In the next theorem, we consider properties P of compactifications that satisfy an additional condition. This condition is satisfied by most of the properties mentioned in [4, §§3.5, 3.10 and 3.13]. In particular, it is satisfied if the property P of compactifications (ψ, X) is that X is semitopological, topological, or a group.

3.7 **Theorem.** *Let N be a closed normal subgroup of G with G/N compact. Suppose that P is a property of compactifications such that $(\psi|_N, \psi(N)^-)$ is a P-compactification of N whenever (ψ, G) is a P-compactification of G. Suppose that the universal P-compactification $(\varepsilon, N^{\mathscr{P}})$ is compatible with G. If $(G\times N^{\mathscr{P}})/\rho$ has property P (as a compactification of G), then $(G\times N^{\mathscr{P}})/\rho \cong G^{\mathscr{P}}$.*

Proof. We show that $(G \times N^{\mathscr{P}})/\rho$ solves the universal mapping problem for P-compactifications of G. Let (ψ, X) be such a compactification. Since $(\psi|_N, \psi(N)^-)$ is a P-compactification of N, it follows from the universal property of $N^{\mathscr{P}}$ that $\psi|_N$ extends to a continuous homomorphism $\tilde{\psi}: N^{\mathscr{P}} \to X$. We first show that

$$(*) \qquad \tilde{\psi}(\sigma_s(x)) = \psi(s^{-1})\tilde{\psi}(x)\psi(s)$$

for all $s \in G$ and $x \in N^{\mathscr{P}}$. Indeed, for fixed $s \in G$, both sides represent homomorphisms of $N^{\mathscr{P}}$ into X; both sides are continuous in x (for σ_s is continuous by compatibility, and the right-hand side by Lemma 2.2); and the two expressions coincide on the dense subspace N (on which $\tilde{\psi} = \psi$).

Now the map $\psi \times \tilde{\psi}: G \times N^{\mathscr{P}} \to X$ defined by

$$\psi \times \tilde{\psi}(s, x) = \psi(s)\tilde{\psi}(x)$$

is continuous (Lemma 2.1) and a homomorphism, since

$$\begin{aligned}\psi \times \tilde{\psi}((s, x)(t, y)) &= \psi \times \tilde{\psi}(st, \sigma_t(x)y) = \psi(st)\tilde{\psi}(\sigma_t(x)y)\\ &= \psi(s)\psi(t)\tilde{\psi}(\sigma_t(x))\tilde{\psi}(y) = \psi(s)\psi(t)\psi(t^{-1})\tilde{\psi}(x)\psi(t)\tilde{\psi}(y)\\ &= \psi(s)\tilde{\psi}(x)\psi(t)\tilde{\psi}(y) = \psi \times \tilde{\psi}(s, x)\psi \times \tilde{\psi}(t, y).\end{aligned}$$

Also $\psi \times \tilde{\psi}$ is constant on ρ-classes, as, if $(s, x)\ \rho\ (t, y)$ so that $t^{-1}s \in N$ and $\varepsilon(t^{-1}s)x = y$, then

$$\begin{aligned}\psi \times \tilde{\psi}(s, x) &= \psi(s)\tilde{\psi}(x) = \psi(t)\psi(t^{-1}s)\tilde{\psi}(x)\\ &= \psi(t)\tilde{\psi} \circ \varepsilon(t^{-1}s)\tilde{\psi}(x) = \psi(t)\tilde{\psi}(\varepsilon(t^{-1}s)x)\\ &= \psi(t)\tilde{\psi}(y) = \psi \times \tilde{\psi}(t, y).\end{aligned}$$

Thus, the quotient of $\psi \times \tilde{\psi}$ gives the required continuous homomorphism

$$(G \times N^{\mathscr{P}})/\rho \to X. \quad \square$$

In some situations, we want to be able to conclude that the right topological semigroup $(G \times X)/\rho$ of 3.6 is also left topological. To reach this conclusion by the methods above, it is necessary to assume as well that the map $s \to \sigma_s(x): G \to X$ is continuous for all $x \in X$ (although, as will be seen later (4.5), we are sometimes able to reach this conclusion with a less restrictive assumption).

3.8 **Lemma.** *Let X be a compactification of N compatible with G. Suppose that $s \to \sigma_s(x)$ is continuous for all $x \in X$. Then*

$$(s, x) \to \sigma_s(x): G \times X \to X$$

is continuous.

Proof. Since $(s, x) \to \sigma_s(x)$ is a group action, this is a consequence of Ellis's Theorem. □

We now deal with some specific compactifications.

3.9 Theorem. *Let N be a closed normal subgroup of G with G/N compact.*

(i) $(G \times N^{\mathscr{LC}})/\rho \cong G^{\mathscr{LC}}$.

(ii) $(G \times N^{\mathscr{D}})/\rho \cong G^{\mathscr{D}}$.

(iii) *If* $s \to \sigma_s(x) \colon G \to N^{\mathscr{WAP}}$ *is continuous for all* $x \in N^{\mathscr{WAP}}$, *then*

$$(G \times N^{\mathscr{WAP}})/\rho \cong G^{\mathscr{WAP}}.$$

(iv) *If* $s \to \sigma_s(x) \colon G \to N^{\mathscr{AP}}$ *is continuous for all* $x \in N^{\mathscr{AP}}$, *then*

$$(G \times N^{\mathscr{AP}})/\rho \cong G^{\mathscr{AP}}.$$

Proof. For each of these we apply Theorem 3.7, noting first that the property P of compactifications (ψ, X) that we are dealing with is, for the various parts:

(i) no further restriction on compactification (ψ, X).

(ii) X is a group.

(iii) X is a semitopological semigroup.

(iv) X is a topological semigroup.

One easily concludes that these properties satisfy the extra condition in the statement of Theorem 3.7, noting for (ii) that a closed subsemigroup S of a compact right topological group X is a group (which follows from the fact that the identity of a maximal subgroup S_1 in the minimal ideal of S will have to be the identity of G, hence $S = S_1$). The universal properties of the compactifications of N mentioned in the statement of the theorem imply that these compactifications are compatible with G. The arguments required are fairly general and will serve to show that many universal compactifications of N are compatible with G; we give the details for (iii). Let $\varepsilon : N \to N^{\mathscr{WAP}}$ be the evaluation map, $\varepsilon(r)(f) = f(r)$ for $r \in N$, $f \in N^{\mathscr{WAP}}$. For $s \in G$, the map $\varepsilon \circ \sigma_s$ is a continuous homomorphism of N into a compact semitopological semigroup. By the universal property of $N^{\mathscr{WAP}}$ [8, or 1, 4.2.11], $\varepsilon \circ \sigma_s$ factors through $N^{\mathscr{WAP}}$: there is a continuous homomorphism $\nu \colon N^{\mathscr{WAP}} \to N^{\mathscr{WAP}}$ such that $\varepsilon \circ \sigma_s = \nu \circ \varepsilon$. ν is the continuous function extending σ_s that is required by the definition of compatibility.

Since $(G \times N^{\mathscr{LC}})/\rho$ is a compactification of G (3.6), the proof of (i) is complete, as is that of (ii) once we note that, since $N^{\mathscr{D}}$ is a group, $(G \times N^{\mathscr{D}})/\rho$, the quotient by a congruence of a semidirect product of groups is also a group.

To finish the proof of (iii), we need only note that, since $N^{\mathscr{WAP}}$ is a semitopological semigroup, so is $G \times N^{\mathscr{WAP}}$ (using 3.8), and also $(G \times N^{\mathscr{WAP}})/\rho$ (2.3(ii)(c)). The proof of (iv) is similar. □

The results of 3.9 have analogues for some "nonsemigroup" compactifications, for example, for the spectra of left translation invariant C^*-subalgebras of $\mathscr{LC}$. $\mathscr{PD}$ and maximal subalgebras $\mathscr{M}$ of $\mathscr{MIN}$ are this kind of subalgebra, and they need not be left m-introverted, so their spectra need not be semigroup compactifications. (We point out that such an algebra $\mathscr{M}$ consists of functions of the form $s \to h(\varepsilon_1(s)\nu)$, where (ε_1, X_1) is the $\mathscr{LC}$-compactification, ν is a fixed idempotent in a fixed minimal left ideal $L \subset X_1$, and $h \in \mathscr{C}(L)$; see [4, §4.8, especially 4.8.3].)

For, let $\mathscr{H} = \mathscr{H}(N)$ be such a subalgebra of $\mathscr{LC}(N)$, with spectrum $X := N^{\mathscr{H}}$ and evaluation map $\varepsilon : N \to X$. (Then $\varepsilon^*(\mathscr{C}(X)) = \mathscr{H}$.) The product of an arbitrary pair of elements of X may be impossible to define in a useful way, but, for $r \in N$ and $x \in X$, $\varepsilon(r)x$ is defined by $\varepsilon(r)x(f) = x(L_r f)$ for all

$f \in \mathscr{H}$ (where $L_r f$ is the *left translate* of f by r, $L_r f(r_1) = f(r_1 r)$). Then $(r, x) \to \varepsilon(r)x \colon N \times X \to X$ is an action ($\varepsilon(rr_1)x = \varepsilon(r)(\varepsilon(r_1)x)$) and (N, X) is a flow (as in [12, 9.2 ff.]). In particular, a multiplication m is induced on $\varepsilon(N)$ and ε is a homomorphism of N onto $\varepsilon(N)$. There is a peculiar aspect of m in the case of $\mathscr{M}$, when $X \cong L$ and $\varepsilon : r \to \varepsilon_1(r)\nu \in L$. m need not agree with the multiplication L has as a subsemigroup of the compactification X_1; $\varepsilon(N)$ need not even be a subsemigroup of L for this latter multiplication! The two multiplications will agree on $\varepsilon(N)$ if N is abelian.

We may now proceed, as above (and in [16]), to define the equivalence relation ρ on $G \times X$, and the conclusions of Lemmas 3.1–3.3 can still be established in this setting. Also the fact that $(s, \varepsilon(r))\ \rho\ (s_1, \varepsilon(r_1))$ and $(t, x)\ \rho\ (t_1, x_1)$ will imply

$$(s, \varepsilon(r))(t, x) := (st, \varepsilon(t^{-1}rt)x)\ \rho\ (s_1 t_1, \varepsilon(t_1^{-1}r_1 t_1)x_1) = (s_1, \varepsilon(r_1))(t_1, x_1).$$

Thus $(s, (t, x)) \to (st, x) \colon G \times ((G \times X)/\rho) \to (G \times X)/\rho$ is an action and

$$(G, Y) := (G, (G \times X)/\rho)$$

is a flow in which (N, X) is canonically embedded via $x \to (e, x)$. (More precisely, x maps to $\{(r, \varepsilon(r^{-1})x) \mid r \in N\}$, the ρ-class of (e, x).)

Here are our analogues of 3.9. We present 3.9(ii) again, as it fits nicely into this context as well.

3.10 **Theorem.** *Let N be a closed normal subgroup of G with G/N compact.*
(i) $(G \times N^{\mathscr{D}})/\rho \cong G^{\mathscr{D}}$.
(ii) $(G \times N^{\mathscr{PD}})/\rho \cong G^{\mathscr{PD}}$.
(iii) $(G \times N^{\mathscr{M}})/\rho \cong G^{\mathscr{M}}$.

Proof. We start with some details for (ii). (The middle author thanks Professor T.-S. Wu for correspondence on these matters some years ago, in particular, for the reference [16].) Let $\varepsilon \colon N \to N^{\mathscr{PD}}$ be the evaluation mapping. Then $N^{\mathscr{PD}}$ is the *universal minimal point distal flow* of N with $1 = \varepsilon(e)$ as distal point. To see this, we show first that the flow $(N, N^{\mathscr{PD}})$ with action

$$(r, x) \to \varepsilon(r)x \colon N \times N^{\mathscr{PD}} \to N^{\mathscr{PD}}$$

is minimal and point distal with 1 as distal point. If the flow is not minimal, there is an $x \in N^{\mathscr{PD}}$ with $\varepsilon(N)x$ not dense in $N^{\mathscr{PD}}$; one sees easily how to get a function $f \in \mathscr{PD}(N) \cong \mathscr{C}(N^{\mathscr{PD}})$, f not equal to the constant function 1, with a net of right translates converging pointwise to 1. But such an f is not in $\mathscr{PD}(N)$, a contradiction. Next, if the flow is not point distal with $1 = \varepsilon(e)$ as distal point, there are $x, x_1 \in N^{\mathscr{PD}}$ and $\{r_\alpha\} \subset N$ with $\varepsilon(r_\alpha) \to x_1$ and $\varepsilon(r_\alpha)x \to x_1$. Let $f \in \mathscr{PD}(N) \cong \mathscr{C}(N^{\mathscr{PD}})$ be 1 at $\varepsilon(e)$ and 0 at x, and let $\{\varepsilon(u_\beta)\}$ converge to x. Then (the pointwise limit) $h \colon = \lim_\beta R_{u_\beta} f$ is not equal to f, but $\lim_\alpha R_{r_\alpha} h = \lim_\alpha R_{r_\alpha} f$, i.e., $f \notin \mathscr{PD}(N)$, the same contradiction.

Now, for the universal property, let (N, X) be a minimal point distal flow with $x \in X$ as distal point. Each function $\tilde{f} \colon r \to f(rx)$ for an $f \in \mathscr{C}(X)$ is in $\mathscr{PD}(N)$; one sees this by applying [29, Proposition 2.1] to $\{f(\cdot\, x_1) \mid x_1 \in X\}$, which is just the pointwise closure of $R_N \tilde{f}$, the set of right translates of $\tilde{f}$. The required map of $N^{\mathscr{PD}}$ onto X is the adjoint of the injection $f \to \tilde{f} \colon \mathscr{C}(X) \to \mathscr{PD}(N)$.

Continuing, we show that $(G, (G \times N^{\mathcal{PD}})/\rho)$ is minimal point distal with (the ρ-class of) $(e, 1)$ as distal point. The minimality follows from that of $(N, N^{\mathcal{PD}})$:

$$\{(str^{-1}, \varepsilon(r)x) \mid s \in G, \ r \in N\}$$

is dense in $G \times N^{\mathcal{PD}}$ for all $(t, x) \in G \times N^{\mathcal{PD}}$. For the point distality, we assume that we have a net $\{t_\alpha\} \subset G$ and (t, x) such that

$$(t_\alpha, (e, 1)) = (t_\alpha, 1) \to (t_1, x_1) \text{ and } (t_\alpha, (t, x)) = (t_\alpha t, x) \to (t_1, x_1).$$

(Convergence here and throughout this proof is in $(G \times N^{\mathcal{PD}})/\rho$ unless specific mention is made to the contrary. Of course, convergence of a net in $G \times N^{\mathcal{PD}}$ implies the convergence of its image, under the quotient map π, in $(G \times N^{\mathcal{PD}})/\rho$.) We must show $(e, 1)\ \rho\ (t, x)$. Let $K \in G$ be compact with $KN = G$, and let each $t_\alpha = s_\alpha r_\alpha$ for an $s_\alpha \in K$ and $r_\alpha \in N$. Then, without loss of generality, we may assume that

$$(t_\alpha, 1)\ \rho\ (t_\alpha r_\alpha^{-1}, \varepsilon(r_\alpha)) = (s_\alpha, \varepsilon(r_\alpha)) \to \text{ some } (t_2, x_2) \in K \times N^{\mathcal{PD}}$$

(convergence in $K \times N^{\mathcal{PD}}$). Similarly,

$$(t_\alpha t, x)\ \rho\ (t_\alpha t u_\alpha^{-1}, \varepsilon(u_\alpha)x) \to \text{ some } (t_3, x_3)$$

(convergence in $K \times N^{\mathcal{PD}}$) for a suitable net $\{u_\alpha\} \subset N$. Of course,

$$(t_1, x_1)\ \rho\ (t_2, x_2)\ \rho\ (t_3, x_3),$$

so $(t_3 r^{-1}, \varepsilon(r)x_3) = (t_2, x_2)$ for some $r \in N$. Now, with $v_\alpha := ru_\alpha$, we have

$$(t_\alpha t v_\alpha^{-1}, \varepsilon(v_\alpha)x) \to (t_2, x_2)$$

in $K \times N^{\mathcal{PD}}$. Thus, $r_\alpha t v_\alpha^{-1} = (t_\alpha r_\alpha^{-1})^{-1}(t_\alpha t v_\alpha^{-1}) \to e$ in G, which implies that $e \in tN$, since N is normal, i.e., $t \in N$. Finally,

$$\varepsilon(v_\alpha t^{-1})1 = \varepsilon(v_\alpha t^{-1} r_\alpha^{-1} r_\alpha) = \varepsilon(v_\alpha t^{-1} r_\alpha^{-1})\varepsilon(r_\alpha) \to x_2$$

and $\varepsilon(v_\alpha t^{-1})(\varepsilon(t)x) \to x_2$ (convergence in $N^{\mathcal{PD}}$ both times), so $\varepsilon(t)x = 1$ and $(e, 1)\ \rho\ (t, x)$, as required.

The method of proof of 3.7 can now be applied to show that $(G \times N^{\mathcal{PD}})/\rho$ has the universal property of $G^{\mathcal{PD}}$, hence must be (isomorphic to) $G^{\mathcal{PD}}$. This completes the proof of (ii).

Proofs of (i) and (iii) can be given along similar lines. However, (i) has been dealt with in 3.9(ii), and we can prove (iii) quickly by citing 4.2(ii) below, which (along with [12, Proposition 7.13 ff.]) gives the result fairly explicitly. □

4. Applications: A structure theorem and function spaces

We now present two (theoretical) applications of our main results. We shall turn to specific groups in the next section. The idea of the first is that, if we can construct the compactification $G^{\mathcal{P}}$ from $N^{\mathcal{P}}$, then we should be able to obtain structural features of $G^{\mathcal{P}}$—such as minimal ideals—from those of $N^{\mathcal{P}}$. The second application says the same for the function spaces associated with the compactifications: $\mathcal{P}(G)$ must be obtainable from $\mathcal{P}(N)$. We begin with a lemma.

4.1 Lemma. *Assume the hypotheses of Theorem* 3.7. *Suppose that the compact group* G/N *with quotient map* $\pi_1 : G \to G/N$ *has property* P, *so that there is a canonical extension* $\varphi: (G \times N^{\mathscr{P}})/\rho \cong G^{\mathscr{P}} \to G/N$. *Then* $\varphi^{-1}(1) = N^{\mathscr{P}}$ *(or, more precisely,* $\varphi^{-1}(1) = \pi(\{e\} \times N^{\mathscr{P}})$, *where* $\pi : G \times N^{\mathscr{P}} \to (G \times N^{\mathscr{P}})/\rho$ *is the quotient map).*

Proof. The extension hypothesis means that $\pi_1 = \varphi \circ \pi \circ \mu$, i.e., that π_1 factors through $G \times N^{\mathscr{P}}$ and $(G \times N^{\mathscr{P}})/\rho$ (notation as in 3.6). So, since $\pi_1^{-1}(1) = N$, it follows that $(\varphi \circ \pi)^{-1}(1)$ is the closure of the union of the ρ-classes of members of $\mu(N)$, i.e., $(\varphi \circ \pi)^{-1}(1) = N \times N^{\mathscr{P}}$. Hence

$$\varphi^{-1}(1) = \pi(N \times N^{\mathscr{P}}) = \pi(\{e\} \times N^{\mathscr{P}}). \quad \square$$

4.2 Theorem. *Let* P *be such that the hypotheses of Theorem* 3.7 *and Lemma* 4.1 *are satisfied, so that* $(G \times N^{\mathscr{P}})/\rho \cong G^{\mathscr{P}}$. *Consider* $N^{\mathscr{P}}$ *to be a subset of* $G^{\mathscr{P}}$. *Let* $K \subset G$ *be compact with* $KN = NK = G$, *and let* $K_0 := \pi \circ \mu(K)$ *be the image of* K *in* $(G \times N^{\mathscr{P}})/\rho$.

(i) *Every idempotent of* $G^{\mathscr{P}}$ *is in* $N^{\mathscr{P}}$. *If the idempotents of* $G^{\mathscr{P}}$ *or of* $N^{\mathscr{P}}$ *form a semigroup, then the two sets of idempotents are isomorphic semigroups.*

(ii) L *is a minimal left ideal of* $G^{\mathscr{P}}$ *if and only if there is a minimal left ideal* L_0 *of* $N^{\mathscr{P}}$ *with* $L = K_0 L_0$. *The corresponding assertion holds for minimal right ideals* $(R = R_0 K_0)$ *and the minimal ideal* $(M = K_0 M_0 K_0)$.

(iii) *Suppose there is a continuous homomorphism* $t \to r(t): G \to N$ *with* $\sigma_t = \sigma_{r(t)}$ *for all* $t \in G$. *Then* $\Lambda(G^{\mathscr{P}}) = K_0 \cdot \Lambda(N^{\mathscr{P}})$.

Proof. (i) The homomorphism φ of Lemma 4.1 must map idempotents to 1, the only idempotent in the group G/N. The conclusion follows easily.

(ii) We apply the Surjectivity Lemma 2.3 with $S = G \times N^{\mathscr{P}}$, $T = G^{\mathscr{P}}$ and $\theta = \pi$, the quotient map from $G \times N^{\mathscr{P}}$ onto $G^{\mathscr{P}}$. The minimal left ideals of $G \times N^{\mathscr{P}}$ are precisely the sets of the form $G \times L_0$ with L_0 a minimal left ideal in $N^{\mathscr{P}}$. The minimal left ideals L of $G^{\mathscr{P}}$ are therefore of the form

$$\begin{aligned}\pi(G \times L_0) &= \pi(G \times \{1\}) \cdot \pi(\{e\} \times L_0) = \pi \circ \mu(G) \cdot L_0 \\ &= \pi \circ \mu(KN) \cdot L_0 = K_0 \cdot (\pi \circ \mu(N)) \cdot L_0 = K_0 L_0.\end{aligned}$$

To apply the same method to right ideals, we must first find the minimal right ideals of $G \times N^{\mathscr{P}}$. So, let $\tilde{R}$ be a minimal right ideal in $G \times N^{\mathscr{P}}$, let $(s, x) \in \tilde{R}$, and let R_1 be a minimal right ideal of $N^{\mathscr{P}}$. Then $R_0 := \sigma_{s^{-1}}(x) R_1$ is also a minimal right ideal of $N^{\mathscr{P}}$ and

$$\{e\} \times R_0 = (s, x)(\{s^{-1}\} \times R_1) \subset \tilde{R},$$

hence $\tilde{R} = (\{e\} \times R_0) \cdot (G \times N^{\mathscr{P}}) = \bigcup_{s \in G}(s, \sigma_s(R_0))$. Sets of the last form are minimal right ideals of $G \times N^{\mathscr{P}}$; one sees this using the fact that each σ_s, being a continuous automorphism of $N^{\mathscr{P}}$ (3.4), maps R_0 onto a minimal right ideal of $N^{\mathscr{P}}$ (2.3(i)(b)). The latter result quoted tells us also that the minimal right ideals of $G^{\mathscr{P}}$ are precisely the sets of the form $R := \pi(\bigcup_{s \in G}(\{s\} \times \sigma_s(R_0)))$. Now let $x \in R_0$ and let $\{r_\alpha\} \subset N$ be such that $\varepsilon(r_\alpha) \to x$ (where ε is the homomorphism of N into $N^{\mathscr{P}}$).Then, since $N^{\mathscr{P}}$ and $G^{\mathscr{P}}$ are compactifications, we get

$$\begin{aligned}\pi(s, \sigma_s(x)) &= \pi(s, 1)\pi\left(e, \lim_\alpha \varepsilon(s^{-1} r_\alpha s)\right) = \lim_\alpha \pi((s, 1)(e, \varepsilon(s^{-1} r_\alpha s))) \\ &= \lim_\alpha \pi(s s^{-1} r_\alpha s, 1) = \left(\lim_\alpha \pi(r_\alpha, 1)\right) \pi(s, 1) = \pi(e, x)\pi(s, 1) = x\pi \circ \mu(s).\end{aligned}$$

Hence $R = \bigcup_s (R_0 \cdot \pi \circ \mu(s)) = R_0 \cdot \pi \circ \mu(G) = R_0 \cdot \pi \circ \mu(NK) = R_0 \cdot \pi \circ \mu(N) \cdot K_0 = R_0 K_0$.

The characterization of the minimal ideal is now easy, for

$$M = LR = K_0 L_0 R_0 K_0 = K_0 M K_0.$$

(iii) Multiplication in $G \times N^{\mathscr{P}}$ is given by

$$(s, x)(t, y) = (st, \sigma_t(x)y).$$

We first prove that $\Lambda(G \times N^{\mathscr{P}}) = G \times \Lambda(N^{\mathscr{P}})$. It is easy to see that $(s, 1) \in \Lambda(G \times N^{\mathscr{P}})$, if $s \in G$. Therefore, if $(s, x) \in \Lambda(G \times N^{\mathscr{P}})$, so also is $(s^{-1}, 1)(s, x) = (e, x)$. This implies that $x \in \Lambda(N^{\mathscr{P}})$. So $\Lambda(G \times N^{\mathscr{P}}) \subseteq G \times \Lambda(N^{\mathscr{P}})$.

To prove the reverse inclusion, we observe that, in the present setting, $\sigma_t(x) \in \Lambda(N^{\mathscr{P}})$ for $t \in G$ and $x \in \Lambda(N^{\mathscr{P}})$ (2.3(ii)(f)), and the map

$$t \to \sigma_t(x) = \sigma_{r(t)}(x) = \varepsilon(r(t)^{-1}) x \varepsilon(r(t))$$

is continuous from G to $\Lambda(N^{\mathscr{P}})$. That $(s, x) \in \Lambda(G \times N^{\mathscr{P}})$ if $x \in \Lambda(N^{\mathscr{P}})$ follows from the joint continuity property of compactifications.

Using 2.3(ii)(f) again, we now conclude that

$$\begin{aligned} \Lambda(G^{\mathscr{P}}) &\supseteq \pi(G \times \Lambda(N^{\mathscr{P}})) = \pi(G \times \{1\}) \cdot \Lambda(N^{\mathscr{P}}) \\ &= \pi \circ \mu(K) \cdot \varepsilon(N) \cdot \Lambda(N^{\mathscr{P}}) = K_0 \cdot \Lambda(N^{\mathscr{P}}) \end{aligned}$$

(as $\varepsilon(N) \subset \Lambda(N^{\mathscr{P}})$). Finally, if $x \in \Lambda(G^{\mathscr{P}})$, take an $s \in G$ with $\pi(s^{-1}, 1)x \in N^{\mathscr{P}}$ (which is possible, G/N being a group). Since $\pi(s^{-1}, 1) \in G^{\mathscr{P}}$, we find

$$\pi(s^{-1}, 1)x \in \Lambda(G^{\mathscr{P}}) \cap N^{\mathscr{P}} \subseteq \Lambda(N^{\mathscr{P}}).$$

So, $x = \pi(s, 1) \cdot \pi(s^{-1}, 1)x \in K_0 \cdot \Lambda(N^{\mathscr{P}})$. □

4.3 *Remarks.* (i) The reader will have noticed that, in 4.2, the maximal groups in the minimal ideal of $G^{\mathscr{P}}$ have not been described in terms of those of $N^{\mathscr{P}}$. This is because the best we seem to be able to do in general is to observe that such a subgroup H of $G^{\mathscr{P}}$ will have the form $H = RL = R_0 K_0 L_0$ (notation as in 4.2), which is not very satisfactory. However, if K is in the (algebraic) center of G or, more generally, if K_0 is in the center of $G^{\mathscr{P}}$, then $H = RL = K_0 H_0 = H_0 K_0 = H_0 K_0 H_0$, where $H_0 := R_0 L_0$ is a maximal subgroup in the minimal ideal of $N^{\mathscr{P}}$. We get the same conclusion in settings where the minimal ideal is a group (e.g., weakly almost periodic compactifications).

(ii) The conditions of 4.2(iii) are satisfied if N is in the center of G, in which case $\sigma_s = 1$ for all $s \in G$, or if G is a direct product $K \times N$ of groups, when $\sigma_{(s, r)} = \sigma_r$.

We now turn to the consideration of functions on N and G. Our real aim is to show how $\mathscr{P}(G)$ can be obtained from $\mathscr{P}(N)$, but it turns out that we can give our results in a more general formulation.

4.4 **Theorem.** *Let N be a closed normal subgroup of G with G/N compact. Let $\mathscr{F}$ and $\mathscr{G}$ be left m-introverted subalgebras of $\mathscr{LC}(N)$ and $\mathscr{LC}(G)$, respectively. Suppose that $G \to G/N$ extends to a continuous homomorphism $G^{\mathscr{G}} \to G/N$. Then the following are equivalent.*

(i) *$N^{\mathscr{F}}$ is compatible with G and $(G \times N^{\mathscr{F}})/\rho = G^{\mathscr{G}}$.*

(ii) $\mathcal{G}|_N = \mathcal{F}$.

(iii) $N^{\mathcal{F}}$ *is a subsemigroup of* $G^{\mathcal{G}}$, *by which we mean that, if* $(\varepsilon, G^{\mathcal{G}})$ *is the* $\mathcal{G}$*-compactification of* G *and* $(\varepsilon_1, N^{\mathcal{F}})$ *is the* $\mathcal{F}$*-compactification of* N, *then there is a topological isomorphism* φ_1 *of* $N^{\mathcal{F}}$ *into* $G^{\mathcal{G}}$ *such that* $\varepsilon|_N = \varphi_1 \circ \varepsilon_1$.

Proof. The equivalence of (ii) and (iii) is easy Gelfand theory. ($\mathcal{F} \cong \mathcal{C}(N^{\mathcal{F}})$ and $\mathcal{G} \cong \mathcal{C}(G^{\mathcal{G}})$, so $N^{\mathcal{F}}$ is a closed subspace of $G^{\mathcal{G}}$ if and only if $\mathcal{G}|_N = \mathcal{F}$. 'Subsemigroup' uses the joint continuity property.)

(i) implies (iii). The (continuous homomorphism) quotient map

$$\pi : G \times N^{\mathcal{F}} \to (G \times N^{\mathcal{F}})/\rho$$

is injective on the compact set $N^{\mathcal{F}} \cong \{e\} \times N^{\mathcal{F}}$, so gives the topological isomorphism of $N^{\mathcal{F}}$ into $G^{\mathcal{G}}$.

(iii) implies (i). First, using (ii), we show that $N^{\mathcal{F}}$ is compatible with G by defining $\sigma_s(x)(f)$ for $s \in G$, $x \in N^{\mathcal{F}}$ and $f \in \mathcal{F} \cong \mathcal{C}(N^{\mathcal{F}})$ by $\sigma_s(x)(f) = x(g \circ \sigma_s|_N)$, where $g \in \mathcal{G}$, $g|_N = f$. Since every such extension g yields a $g \circ \sigma_s$ agreeing with $f \circ \sigma_s$ on N, hence on $N^{\mathcal{F}}$, $\sigma_s(x)$ is well defined. (Bear in mind, as well, that left m-introversion implies translation invariance.) So $G \times N^{\mathcal{F}}$ is a semigroup (3.5),

$$(s, x)(t, y) = (st, \sigma_t(x)y).$$

Next, let $K \subset G$ be compact with $G = KN$, and identify $N^{\mathcal{F}}$ with its image in $G^{\mathcal{G}}$, so in particular $\varepsilon(r) = \varepsilon_1(r)$ for $r \in N$. If $\nu: G \times N^{\mathcal{F}} \to G^{\mathcal{G}}$ is defined by $\nu(s, x) = \varepsilon(s)x$, then the argument proving $(*)$ in 3.7 shows that $\nu(e, \sigma_s(x)) = \varepsilon(s^{-1})x\varepsilon(s)$ for all $s \in G$ and $x \in N^{\mathcal{F}}$. Hence, ν is a homomorphism and

$$\nu(G \times N^{\mathcal{F}}) = \nu(K \times \{1\})\nu(N \times \{1\})N^{\mathcal{F}} = \nu(K \times \{1\})N^{\mathcal{F}}$$

is a compact subset of $G^{\mathcal{G}}$ (by the joint continuity property of $G^{\mathcal{G}}$) and contains the dense subset $\varepsilon(G)$, so equals $G^{\mathcal{G}}$.

We complete the proof that (iii) implies (i) by showing that ν is just the quotient map of the relation ρ. Observe that, if $\varphi : G^{\mathcal{G}} \to G/N$ is the continuous homomorphism provided by the hypotheses, then $\varphi(N^{\mathcal{F}}) = \varphi((\varepsilon(N))^-) = \{1\}$. Therefore, if $s \in G$ and $\varepsilon(s)x = y$ for some $x, y \in N^{\mathcal{F}}$, we have $\varphi \circ \varepsilon(s) = \varphi \circ \varepsilon(s)\varphi(x) = \varphi(y) = 1$, so that $\varepsilon(s) \in \varepsilon(G) \cap \varphi^{-1}(1) = \varepsilon(N)$, i.e., $s \in N$. Now, the following four assertions about a pair of points $(s, x), (t, y) \in G \times N^{\mathcal{F}}$ are equivalent:

$\nu(s, x) = \nu(t, y)$;

$\varepsilon(s)x = \varepsilon(t)y$;

$\varepsilon(t^{-1}s)x = y$ and $t^{-1}s \in N$; and

$(s, x)\ \rho\ (t, y)$. □

Using Theorem 4.4, we get the following variants of Theorem 3.9, (iii) and (iv).

4.5 **Corollary.** *Let* N *be a closed normal subgroup of* G *with* G/N *compact.*

(i) $(G \times N^{\mathcal{WAP}})/\rho \cong G^{\mathcal{WAP}}$ *if and only if* $\mathcal{WAP}(G)|_N = \mathcal{WAP}(N)$.

(ii) $(G \times N^{\mathcal{AP}})/\rho \cong G^{\mathcal{AP}}$ *if and only if* $\mathcal{AP}(G)|_N = \mathcal{AP}(N)$.

For our main result about functions we need a lemma, the ideas for which go back at least to Ptak [27].

4.6 **Lemma.** *Let X and Y be compact spaces, and let D be a dense subset of Y. Then $f\colon X \times D \to \mathbb{C}$ has a continuous extension $\bar{f}\colon X \times Y \to \mathbb{C}$ if and only if both*

(i) *for each $x \in X$, $d \to f(x, d)$ has a continuous extension to Y, and*
(ii) *$\{f(\cdot, d) \mid d \in D\}$ is an equicontinuous subset of $\mathscr{C}(X)$.*

Proof. If $\bar{f}$ exists, then $d \to \bar{f}(x, d)$ on Y extends $d \to f(x, d)$ on D, so (i) holds. Also, $\{\bar{f}(\cdot, y) \mid y \in Y\}$ is a compact subset of $\mathscr{C}$, so equicontinuous; $\{f(\cdot, d) \mid d \in D\}$ is a subset of this.

Conversely, if $\{f(\cdot, d) \mid d \in D\}$ is equicontinuous, then it has norm compact closure in $\mathscr{C}(X)$ (Arzela-Ascoli). Therefore, if $\{d_\alpha\} \subset D$ and $d_\alpha \to y \in Y$, $\{f(\cdot, d_\alpha)\}$ clusters at a function $\bar{f}(\cdot, y)$ (say) in $\mathscr{C}(X)$. But for each $x \in X$, (i) assures us that $\{f(x, d_\alpha)\}$ converges to a limit which must be $\bar{f}(x, y)$. Thus $f(\cdot, d_\alpha) \to \bar{f}(\cdot, y)$ uniformly. It is easy to see now that $\bar{f}$ is well defined on $X \times Y$, extends f, and is continuous. □

4.7 **Theorem.** *Let N be a closed normal subgroup of G with G/N compact. Suppose that the hypotheses of Theorem 4.4 hold and that $N^{\mathscr{F}}$ is a subsemigroup of $G^{\mathscr{G}}$ (and therefore that* (i), (ii) *and* (iii) *of 4.4 hold). Let $K \subset G$ be compact with $KN = G$. Write $\nu\colon K \times N^{\mathscr{F}} \to G^{\mathscr{G}}$ for the quotient map $K \times N^{\mathscr{F}} \to (K \times N^{\mathscr{F}})/\rho \cong G^{\mathscr{G}}$.*

Let $f\colon K \times N \to \mathbb{C}$ be continuous. Then $f = g \circ \nu$ for some $g \in \mathscr{G}$ if and only if the following three conditions hold:

(i) *f is constant on ρ-classes;*
(ii) *for each $s \in K$, $f(s, \cdot) \in \mathscr{F}$; and*
(iii) *$\{f(\cdot, r) \mid r \in N\}$ is equicontinuous.*

Proof. Start with $g \in \mathscr{G}$. Let $\bar{g}$ be its continuous extension to $G^{\mathscr{G}}$ (i.e., let $\bar{g} \in \mathscr{C}(G^{\mathscr{G}})$ be such that $\varepsilon^*(\bar{g}) = g$). Then $\bar{f} := \bar{g} \circ \nu$ extends $f := g \circ \nu$ to $K \times N^{\mathscr{F}}$. Since $f = g \circ \nu$, (i) holds. Since f extends to $K \times N^{\mathscr{F}}$, (ii) and (iii) come from Lemma 4.6.

Conversely, suppose that (i), (ii) and (iii) hold. Lemma 4.6 tells us that f has a continuous extension $\bar{f}$ to $K \times N^{\mathscr{F}}$. If we prove that $\bar{f}$ is constant on ρ-classes, it will follow that $\bar{f} = \bar{g} \circ \nu$ for some $\bar{g} \in \mathscr{C}(G^{\mathscr{G}})$ and hence that $f = g \circ \nu$ for some $g \in \mathscr{G}$. So take $(s, x)\ \rho\ (t, y)$ in $K \times N^{\mathscr{F}}$, so that $t^{-1}s \in N$ and $\varepsilon_1(t^{-1}s)x = y$. (As in 4.4, $\varepsilon_1\colon N \to N^{\mathscr{F}}$ is the evaluation mapping.) Find $\{r_\alpha\} \subset N$ with $\varepsilon_1(r_\alpha) \to x$. Then

$$x_\alpha := \varepsilon_1(t^{-1}sr_\alpha) \to \varepsilon_1(t^{-1}s)x.$$

By definition, $(s, \varepsilon_1(r_\alpha))\ \rho\ (t, x_\alpha)$. By hypothesis (i), $f(s, \varepsilon_1(r_\alpha)) = f(t, x_\alpha)$. By continuity, $\bar{f}(s, x) = \bar{f}(t, y)$, as required. □

From 4.7 and 3.10, with the help of 4.4, we get

4.8 **Theorem.** *Let N be a closed normal subgroup of G with G/N compact. Let $K \subseteq G$ be compact with $KN = G$. Define $\tau\colon K \times N \to G$ by $\tau(s, r) = sr$. Let $g\colon K \times N \to \mathbb{C}$ be a continuous function that is constant on ρ-classes, so that $g = f \circ \tau$ for some continuous $f\colon G \to \mathbb{C}$.*

(i) *$f \in \mathscr{L}\mathscr{C}(G)$ if and only if $g(s, \cdot) \in \mathscr{L}\mathscr{C}(N)$ for each $s \in K$ and $\{g(\cdot, r) \mid r \in N\}$ is equicontinuous.*

(ii) *Suppose that* $\mathscr{WAP}(G)|_N = \mathscr{WAP}(N)$. *Then* $f \in \mathscr{WAP}(G)$ *if and only if* $g(s, \cdot) \in \mathscr{WAP}(N)$ *for each* $s \in K$ *and* $\{g(\cdot, r) \mid r \in N\}$ *is equicontinuous.*

(iii) *Suppose that* $\mathscr{AP}(G)|_N = \mathscr{AP}(N)$. *Then* $f \in \mathscr{AP}(G)$ *if and only if* $g(s, \cdot) \in \mathscr{AP}(N)$ *for each* $s \in K$ *and* $\{g(\cdot, r) \mid r \in N\}$ *is equicontinuous.*

(iv) $f \in \mathscr{D}(G)$ *if and only if* $g(s, \cdot) \in \mathscr{D}(N)$ *for each* $s \in K$ *and* $\{g(\cdot, r) \mid r \in N\}$ *is equicontinuous.*

(v) $f \in \mathscr{PD}(G)$ *if and only if* $g(s, \cdot) \in \mathscr{PD}(N)$ *for each* $s \in K$ *and* $\{g(\cdot, r) \mid r \in N\}$ *is equicontinuous.*

(vi) $f \in \mathscr{MIN}(G)$ *if and only if there is a maximal subalgebra* $\mathscr{M} \subset \mathscr{MIN}(N)$ *with* $g(s, \cdot) \in \mathscr{M}$ *for each* $s \in K$, *and also* $\{g(\cdot, r) \mid r \in N\}$ *is equicontinuous.*

Proof. These follow from the theorems cited. □

Another characterization of these functions on $G = KN$ can be given in terms of vector-valued functions on N. For example, let $f \in \mathscr{LC}(G)$ and define g on $K \times N$ by $g(s, r) = f(sr)$, i.e., $g = f \circ \tau$, as above. Define $F\colon N \to \mathscr{C}(K)$ by $F(r) = g(\cdot, r)$. As in [4, 4.4.21], we say that $F \in \mathscr{LC}(N, \mathscr{C}(K))$ if

$$(1) \quad \|L_{r_\alpha}F - L_{r_1}F\| := \sup_{r\in N} \| F(r_\alpha r) - F(r_1 r) \| := \sup_{r\in N,\ s\in K} |F(r_\alpha r)(s) - F(r_1 r)(s)| \to 0 \quad \text{whenever } r_\alpha \to r_1.$$

But the absolute value above is just $|f(sr_\alpha r) - f(sr_1 r)|$, which tends to 0 uniformly in s and r, since $sr_\alpha r(sr_1 r)^{-1} \to e$ uniformly in s and r. (The last claim holds, since G is a topological group; then the second last holds, since $f \in \mathscr{LC}(G)$.) Thus $F \in \mathscr{LC}(N, \mathscr{C}(K))$.

We define three spaces of continuous functions from N into $\mathscr{C}(K)$. To begin, the range of each function F considered is norm relatively compact in $\mathscr{C}(K)$ (equivalently, by Arzela-Ascoli, $\{F(r) \mid r \in N\}$ is bounded and equicontinuous); hence, F is norm bounded. The definitions are completed as in [4].

(i) $F \in \mathscr{LC}(N, \mathscr{C}(K))$ if (1) above is satisfied.

(ii) $F \in \mathscr{WAP}(N, \mathscr{C}(K))$ if the set of right translates $R_N F := \{R_r F \mid r \in N\}$ of F is weakly relatively compact in $\mathscr{C}(N, \mathscr{C}(K))$.

(iii) $F \in \mathscr{AP}(N, \mathscr{C}(K))$ if $R_N F$ is norm relatively compact in $\mathscr{C}(N, \mathscr{C}(K))$.

Analogues of these definitions and of the next result can be devised for $\mathscr{D}$, $\mathscr{PD}$ and $\mathscr{MIN}$, and for other spaces. The devoted reader will devise some of these.

4.9 **Theorem.** *Let* G, N, K, τ, g, *and* f *be as in Theorem* 4.8. *Define* $F\colon N \to \mathscr{C}(K)$ *by* $F(r) = g(\cdot, r)$.

(i) $f \in \mathscr{LC}(G)$ *if and only if* $F \in \mathscr{LC}(N, \mathscr{C}(K))$.

(ii) $f \in \mathscr{WAP}(G)$ *if and only if* $F \in \mathscr{WAP}(N, \mathscr{C}(K))$.

(iii) $f \in \mathscr{AP}(G)$ *if and only if* $F \in \mathscr{AP}(N, \mathscr{C}(K))$.

Proof. (i) The proof in one direction is given above. Suppose that

$$F \in \mathscr{LC}(N, \mathscr{C}(K))$$

and $s \in K$. Since $g(s, r) = F(r)(s) = f(sr)$, (1) and the sentence following show directly that $g(s, \cdot) \in \mathscr{LC}(N)$, so $f \in \mathscr{LC}(G)$, by 4.8(i).

(ii) If $f \in \mathscr{WAP}(G)$ and satisfies Grothendieck's double limit criterion (DLC) [15, or 4, p. 139] for weak almost periodicity on G, it follows immediately that each $g(s, \cdot)$ satisfies DLC on N, so $F \in \mathscr{WAP}(N, \mathscr{C}(K))$ (4.8(ii)).

With a small amount of extra work, the converse is a consequence of Theorem 3 in [23]. Put $X = \mathscr{C}(K)$ and $C = N$, and let D be the evaluation functionals on $\mathscr{C}(K)$ by members of K in the cited theorem. If $F \in \mathscr{WAP}(N, \mathscr{C}(K))$, DLC implies that each $g(s, \cdot) \in \mathscr{WAP}(N)$, hence $f \in \mathscr{WAP}(G)$ (4.8(ii)).

(iii) It is easy to verify directly that $f \in \mathscr{AP}(G)$ implies that

$$F \in \mathscr{AP}(N, \mathscr{C}(K)).$$

The converse follows from 4.8(iii). □

5. Examples

We begin by looking at some specializations of our main results.

5.1. The most awkward hypotheses in our theorems are that compactifications of N should be compatible with G and that $s \to \sigma_s$ should be (in some sense) continuous. These conditions are trivially satisfied if σ_s is the identity for all $s \in G$. This is equivalent to

(i) $N \subset Z(G)$, the algebraic centre of G, and it is therefore obviously satisfied if

(ii) G is commutative. Under condition (i), for any property P, $(G \times N^{\mathscr{P}})/\rho \cong G^{\mathscr{P}}$ (Theorem 3.7), including the cases dealt with in Theorems 3.9 and 3.10. If the compactification $G \to G/N$ has property P, then each function in $\mathscr{P}(N)$ extends to a function in $\mathscr{P}(G)$ (Theorem 4.4). By Theorem 4.2, minimal ideals of $G^{\mathscr{P}}$ (etc.) can be obtained from minimal ideals of $N^{\mathscr{P}}$ (etc.); as noted in Remark 4.3(i), we also get

$$\Lambda(G^{\mathscr{P}}) = K_0 \cdot \Lambda(N^{\mathscr{P}}) = \Lambda(N^{\mathscr{P}}) \cdot K_0.$$

5.2. Another special case is when G/N is finite. In this situation it is usually easy to extend functions on N to functions on G just by taking left translates onto the individual cosets. Theorem 4.2 then tells us that $(G \times N^{\mathscr{P}})/\rho \cong G^{\mathscr{P}}$. This holds for the cases dealt with in Theorems 3.9 and 3.10. Theorem 4.6 says that a function $f : G \to \mathbb{C}$ is in $\mathscr{P}(G)$ if and only if $r \to g(sr)$ is in $\mathscr{P}(N)$ for each $s \in G$ (that is, g is the left translate of a $\mathscr{P}(G)$ function on each coset) and $\{s \to g(sr) \mid r \in N\}$ is equicontinuous on some compact set K with $KN = G$; since G/N is finite, we can take K finite, and this latter condition is vacuous.

An example of this case is obtained by taking K to be a finite subgroup of the circle group $\mathbb{T}$ and $N = \mathbb{C}$, and G to be the semidirect product $\mathbb{C} \times K$ (a subgroup of the euclidean group of the plane $\mathbb{C} \times \mathbb{T}$, as in 5.5).

5.3. Let us take a very specific example, $G = \mathbb{R}$, $N = \mathbb{Z}$. If we take $K = [0, 1] \subset \mathbb{R}$, then $K + N = G$. Since G is abelian, we are in the case 5.1(ii).

We show how our theorems apply by looking at the $\mathscr{LC}$-compactification. Since $\mathbb{Z}$ is discrete, $\mathbb{Z}^{\mathscr{LC}} = \beta\mathbb{Z}$. If we regard $\mathbb{Z}$ as a subset of $\beta\mathbb{Z}$, then $1 + x$ and $x + 1$ are defined for each $x \in \beta\mathbb{Z}$ and $1 + x = x + 1$. (Here $1 \in \mathbb{Z}$, of course.) The equivalence relation ρ on $[0, 1] \times \beta\mathbb{Z}$ is given by $(s, x)\ \rho\ (t, y)$ if and only if $-t + s \in \mathbb{Z}$ and $-s + t + x = y$. Since $s, t \in [0, 1]$, this means that either $(s, x) = (t, y)$, or $s = 1$, $t = 0$ and $y = x + 1$, or $s = 0$, $t = 1$ and $x = y + 1$. So the only nontrivial equivalence classes are of the form $\{(1, x), (0, x + 1)\}$. This says that $\mathbb{R}^{\mathscr{LC}}$ is obtained from $\beta\mathbb{Z}$ by adjoining a unit interval between each pair of points x and $x+1$ in $\beta\mathbb{Z}$ (just as $\mathbb{R}$ itself can be obtained from $\mathbb{Z}$).

According to 5.1, minimal left ideals in $\mathbb{R}^{\mathcal{LC}}$ are obtained by taking minimal left ideals in $\beta\mathbb{Z}$ and adjoining a unit interval between x and $x+1$ for each x. The same is true of $\Lambda(\mathbb{R}^{\mathcal{LC}})$; but $\Lambda(\beta\mathbb{Z})$ is $\mathbb{Z}$, so $\Lambda(\mathbb{R}^{\mathcal{LC}})$ is $\mathbb{R}$. It follows too that $\mathbb{R}^{\mathcal{LC}}$ has precisely 2^c minimal left ideals and 2^c minimal right ideals, and contains a copy of the free group on 2^c generators, for these facts are true for $\beta\mathbb{Z}$ (as for $\beta\mathbb{N}$ [19]).

Now let g be a function on $\mathbb{R}$. For g to be in $\mathcal{LC}(\mathbb{R})$, it is necessary and sufficient that $n \to g(t+n)$ is in $\mathcal{LC}(\mathbb{Z})$ for each $t \in [0, 1]$ (a vacuous condition, since $\mathcal{LC}(\mathbb{Z})$ consists of all bounded functions on $\mathbb{Z}$) and that $\{t \to g(t+n) \mid n \in \mathbb{Z}\}$ is equicontinuous on [0, 1] (Theorem 4.8).

Similar results hold for other compactifications of $\mathbb{R}$. For example,

$$\mathbb{R}^{\mathcal{WAP}} = ([0, 1] \times \mathbb{Z}^{\mathcal{WAP}})/\rho.$$

The recent results of Ruppert [28] on idempotents in $\mathbb{Z}^{\mathcal{WAP}}$ therefore hold for $\mathbb{R}$, too. His constructions of functions in $\mathcal{WAP}(\mathbb{Z})$ can be used to produce functions in $\mathcal{WAP}(\mathbb{R})$. If $g \in \mathcal{WAP}(\mathbb{Z})$, write $x \in \mathbb{R}$ in the form $x = t+n$ with $t \in [0, 1]$, define

$$f(x) = (1-t)g(n) + tg(n+1),$$

and apply Theorem 4.8.)

Analogues of these conclusions about $\mathbb{R} = [0, 1] + \mathbb{Z}$ hold for $\mathbb{R}^n = [0, 1]^n + \mathbb{Z}^n$.

5.4. Here is a trivial example. If K and N are locally compact groups with K compact, then the direct product $G = K \times N$ satisfies our conditions.

5.5. The euclidean group (or motion group) of the plane is an interesting example, $G = \mathbb{C} \times \mathbb{T}$ with

$$(z, w)(z_1, w_1) = (z + wz_1, ww_1).$$

Here the normal subgroup $N = \mathbb{C} \times \{1\} \cong \mathbb{C}$. Then $\mathcal{AP}(G)|_N \subsetneq \mathcal{AP}(N)$ [6, 24], so $G^{\mathcal{AP}} \ncong (G \times \mathbb{C}^{\mathcal{AP}})/\rho$ (4.5(ii)). $\mathbb{C}^{\mathcal{AP}}$ is compatible with G, so it is the continuity of $s \to \sigma_s$, as required in 3.9(iv), that must fail to hold. From [6, 24], we can similarly conclude that $G^{\mathcal{WAP}} \ncong (G \times \mathbb{C}^{\mathcal{WAP}})/\rho$, and that the continuity of $s \to \sigma_s$, as required by 3.9(iii), also fails to hold.

We exhibit two compactifications of $\mathbb{C}$ that are not compatible with G.

(i) Let the compactification $(\psi, \mathbb{T})$ of $\mathbb{C}$ be given by $\psi(z) = \psi(x+iy) = e^{ix}$. For compatibility, we need the convergence of $\{e^{ix_n}\} = \{(e^i)^{x_n}\}$ in $\mathbb{T}$ to imply the convergence of

$$\{\sigma_{(z,w)}(\psi(x_n))\} = \{\psi((-z/w, w^{-1})(x_n, 1)(z, w))\} = \{(e^{i\cos\varphi})^{x_n}\},$$

where $w = e^{i\varphi}$. Kronecker's Theorem [18] says explicitly that this will not happen if e^i and $e^{i\cos\varphi}$ are linearly independent, i.e., if $\cos\varphi \in \mathbb{R} \setminus \mathbb{Q}$.

(ii) We can modify the compactification in (i) and get one that is injective on $\mathbb{C}$. The left m-introverted subalgebra $\mathcal{F}$ of $\mathcal{LC}(\mathbb{C})$ that $(\psi, \mathbb{T})$ corresponds to (i.e., $\psi^*(\mathcal{C}(\mathbb{T}))$ is just the set of $\mathbb{C}$-valued functions $x + iy \to f(x)$, where f is continuous and has period 2π. The direct sum $\mathcal{F} \oplus \mathcal{C}_0(\mathbb{C})$ is also left m-introverted, and its resulting compactification is injective and not compatible with G. ($\mathcal{C}_0$ is the algebra of continuous functions vanishing at infinity.)

The existence of noncompatible compactifications like these does not impede the conclusions $(G \times \mathbb{C}^{\mathcal{LC}})/\rho \cong G^{\mathcal{LC}}$, $(G \times \mathbb{C}^{\mathcal{D}})/\rho \cong G^{\mathcal{D}}$, $(G \times \mathbb{C}^{\mathcal{PD}})/\rho \cong G^{\mathcal{PD}}$, and $(G \times \mathbb{C}^{\mathcal{M}})/\rho \cong G^{\mathcal{M}}$ (3.9 and 3.10), even though

$$\mathcal{F} \subset \mathcal{D}(\mathbb{C}) \subset \mathcal{PD}(\mathbb{C}) \subset \mathcal{M}(\mathbb{C}) \subset \mathcal{LC}(\mathbb{C})$$

and $\mathcal{F} \oplus \mathcal{C}_0(\mathbb{C}) \subset \mathcal{LC}(\mathbb{C})$. We pose a question: if $(G \times N^{\mathcal{P}})/\rho \ncong G^{\mathcal{P}}$ for some property P, does it follow that there is a compactification (ψ, X) of N with $\psi^*(\mathcal{C}(X)) \subset \mathcal{P}(N)$ that is not compatible with G?

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